DISPERSE: A GENERAL PURPOSE PROGRAM FOR CREATING DISPERSION CURVES

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INTRODUCTION

The application of guided waves in NDT can be hampered by the lack of readily available dispersion curves for complex structures. To overcome this hindrance, we have developed a general purpose program that can create dispersion curves for a very wide range of systems and then effectively communicate the information contained within those curves. The program uses the global matrix method to handle multi-layered Cartesian and cylindrical systems. The solution routines cover both leaky and non-leaky cases and remain robust for systems which are known to be difficult, such as large frequency-thicknesses and thin layers embedded in much thicker layers. Elastic and visco-elastic isotropic materials are fully supported; anisotropic materials are also covered, but are currently limited to the elastic, non-leaky, Cartesian case.

An extremely large amount of work has already been done to describe the wave propagation in layered systems, which began in the late nineteenth century and continues today There is not enough space in this paper to describe the contributions that many excellent researchers have made to the field. Instead, this paper describes how we have combined some of this previous work and our own research to create a robust, user friendly, general purpose tool. A review of matrix techniques as they apply to modelling ultrasonic waves in multi-layered media is given in [1].

When creating dispersion curves, the displacements and stresses for each type of material and geometry are described in a material layer matrix. By satisfying the given boundary conditions at each interface, the individual layer matrices are assembled to describe the behaviour of the entire system. The dispersion curves then emerge as solutions to the assembled system of equations.

Once the dispersion curves are generated, our program provides many easy methods to explore and use the information contained within the curves. A mode shape display continually updates the distribution of stresses, displacements, and energy as points are selected on a dispersion curve. The phase velocity, group velocity, attenuation, real wave-number, angle of incidence, etc. can be shown and compared. In addition, an interface to our finite element program allows the interaction of guided waves with defects to be examined.

The program has been developed as a by-product of our research. It has proven to be a valuable tool for understanding wave propagation in complex systems and transferring that understanding to new ultrasonic testing applications

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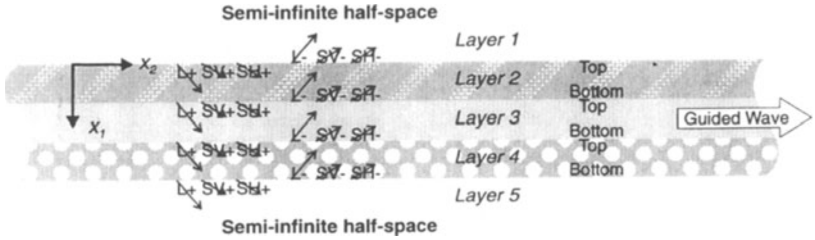


Figure 1 The definition of axes and geometry used in this paper. The example shows a representative sample with three finite layers and two infinite half-spaces. The partial waves (L+-,SV+-,SH+-) combine to form a guided wave.

GLOBAL MATRIX METHOD

To create dispersion curves, a general purpose program must first describe the propagation of waves in a bulk material and relate that information to the boundary conditions at an interface Then, individual layers may be assembled to represent a complete system.

Creating the Layer Matrix

The core of our general purpose approach is a layer matrix which relates the stresses and displacements in each of the layers to the partial waves in the material. As shown in Figure 1, each of the interior layers has six partial waves, a longitudinal wave (L+), a vertically polarized shear wave (SV+) and a horizontally polarized shear wave (SH+) propagating down from the top of the layer and three more waves propagating up from the bottom of the layer (L-, SV-, SH-). With appropriate orientation, amplitude, and phase, the superposition of these six partial waves can describe any field, provided the system is linear. The semi-infinite half-spaces, layers 1 and 5, only need 3 partial waves since we assume that no energy is being added to the system at infinity. Hooke's law, Euler's equation of motion, and Navier's displacement equation of motion provide expressions for the displacements and the stresses at anywhere in the layer as a function of the partial waves. For examples of this derivation for two isotropic cases, see [1] and [2].

At a perfect interface between two ideal solids, the displacements are continuous. In addition, the normal stress across the interface and the two shear stresses along the interface are continuous. Similarly, for a perfect liquid-liquid or liquid-solid boundary, the normal displacement and normal stress are continuous. Specifying these quantities in terms of the amplitudes of the six partial waves for each layer yields a six by six layer matrix, which changes for each type of material and each geometry. Currently, elastic and visco-elastic isotropic materials are fully incorporated in the program for both Cartesian and cylindrical geometries, provided the mode propagates axially down the pipe, not around it. In addition, the program supports vacuum-loaded elastic anisotropic flat plates, as well as limited cases of waves propagating around the circumference of an isotropic pipe.

Assembling the Layer Matrices Into a Global Matrix

Once we know how to describe each material layer and the boundary conditions for each, we are ready to combine the layers to describe the entire system. We have settled on the global matrix method, which was first proposed by Knopoff in 1964[3] and subsequently developed by other researchers such as Randall[4] and Schmidt[5] Compared to the Thomson-Haskell transfer matrix technique[6, 7], this method has the advantage that it remains stable at high frequency-thickness products. In addition, the global matrix method also allows the same

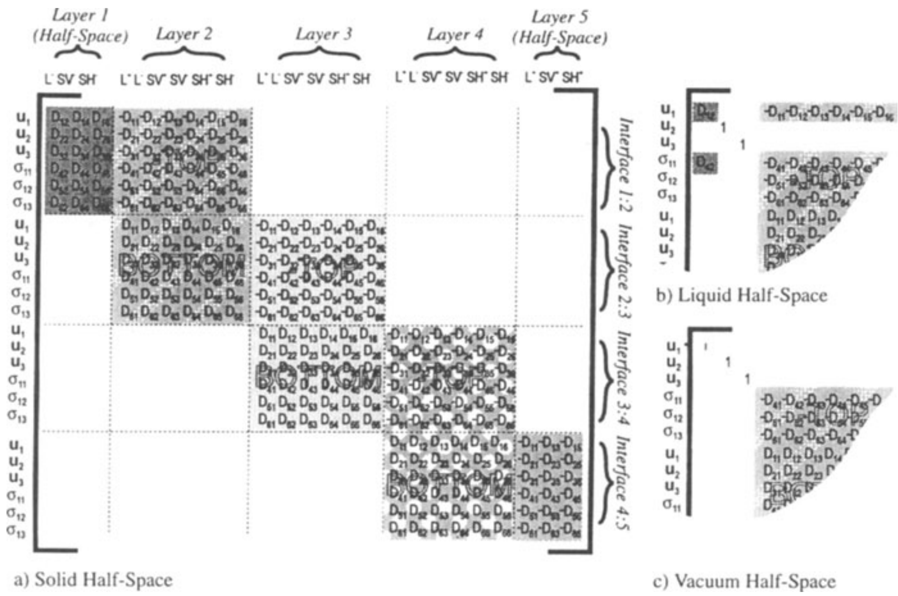


Figure 2. The structure of the global matrix for (a) solid, (b) liquid, and (c) vacuum half-space, where blank spaces are zeros, $\mathbf{D}_{ij}$ is the layer matrix, and L+-, SV+-, and SH+- are the partial waves in the various layers.

intuitive base matrix to be used for real or complex wavenumber, vacuum, liquid or solid half-spaces, and modal or response solutions. The disadvantage is that the global matrix may be large and the solution therefore may be relatively slow when the systems involve many layers. However, the speed of modern computers reduces the effect of this limitation. A comparison and description of various matrix techniques can be found in [1].

In the global matrix method, a single matrix represents the complete system. The global matrix consists of $6(n-1)$ equations, where n is the number of layers (including each of the semi-infinite half-spaces as a layer). The equations are based, in sets of six, on satisfying the boundary conditions at each interface. The columns of the global matrix correspond to the amplitudes of the partial waves in each layer, six for each of the interior layers and three for each of the exterior (semi-infinite) layers. Figure 2 shows how we assemble the global matrix. If one or both of the half-spaces is either a liquid or a vacuum, the matrix can be easily modified to remove the unnecessary boundary conditions, as demonstrated for the 'top' half space in Figures 2(b) and 2(c). Solutions to the wave propagation problem correspond to non-trivial solutions of the characteristic equation, $[\mathbf{G}]\{\mathbf{A}\} = 0$, where $[\mathbf{G}]$ the global matrix, and $\{\mathbf{A}\}$ is a vector of the partial wave amplitudes.

FINDING A ROOT

In order to find a point on a dispersion curve, we need to find a root of the characteristic equation, which in this case corresponds to a point where the absolute value of the determinant of the global matrix is zero. In general, the frequency, real wave number, and attenuation can all be varied to find the roots. In practice, to find a root, we perform a coarse sweep, for which two of the variables are held constant while the third is varied. If a minimum of the characteristic equation is found, the routine begins a fine search, which then tries to converge onto a valid root.

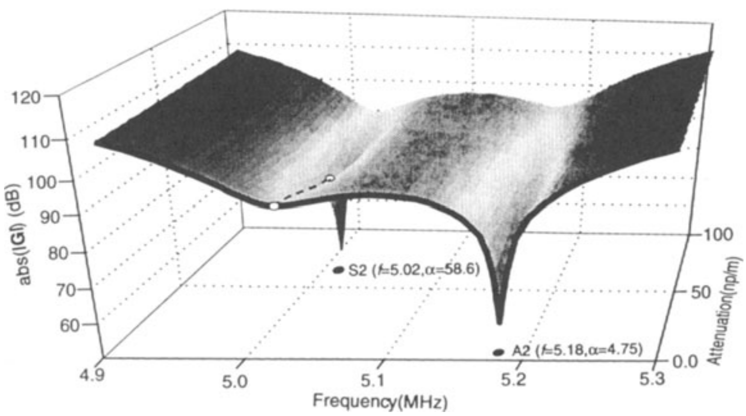


Figure 3. The process of finding a root involves a coarse sweep (thick line) to find an initial minimum and a fine search (dashed lines) to narrow down on a root (solid circle). The example above shows the absolute value of the determinant of the global matrix for a 1 mm steel plate immersed in water at a wave number of 4 rad/mm.

If all of the materials are elastic and the half-spaces are both vacuum, then there is no way for energy to leave the system and so there will be no attenuation. In this case the fine search routine can be relatively simple. Since the attenuation will always be zero, a single one dimensional minimum search routine suffices to find the root. However, the process is more complicated when attenuation exists and a two dimensional search routine is required. The attenuation is an important property of the system, describing the rate at which a guided mode decays due to damping or leakage as it travels. It is unknown and has to be found as part of the solution.

Figure 3 illustrates the process of finding a valid root of the characteristic equations when attenuation is present in the system. The surface corresponds to the log of the absolute value of the determinant of the global matrix when the real wave number is held at 4 rad/mm for a 1 mm steel plate immersed in water. To find a root, a coarse frequency sweep is made at zero attenuation, as shown by the thick line at the front of the surface. The two minima are identified as possible roots and are used as starting points for a fine search. The left minimum is less well defined than the right minimum because the attenuation value of the sweep is very different than the actual attenuation of the root.

Next, starting at a minimum located by the coarse sweep, the fine search uses a two dimensional iterative bisection routine (dashed line with open circles) to converge onto the root (solid circle). The routine starts by taking small steps in frequency (dashed line) until a tighter minimum (open circle) is found. The segments that bracket the minimum are each split in two and the function is calculated at these points, allowing a new minimum to be chosen from the middle three points. The bracketing continues until the desired tolerance is achieved, at which point the routine uses the same logic to converge in attenuation. The routine repeats the search for minima in frequency and then attenuation until it determines that it has found a valid root by examining nearby phase changes. A more complete description of this search routine, which we have found to be very robust, can be found in [8].

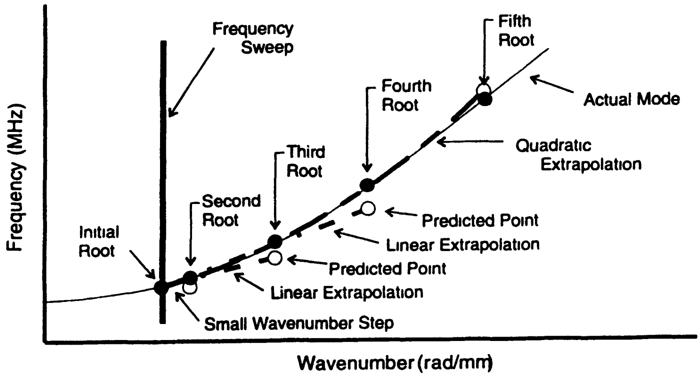


Figure 4. Using an extrapolating routine to trace modes dramatically improves the program's speed and reliability.

TRACING A DISPERSION CURVE

Although it is possible to find a lot of roots of the dispersion equation and connect the dots, there is a much more efficient and robust method for tracing dispersion curves. As shown in Figure 4, the tracing routine starts from a root that was found from a coarse sweep, labelled the initial root. Next the routine takes a very small step in one of the three independent variables, in this case wave-number, and converges on a second root. These first two points are used to linearly extrapolate to and converge upon a third root, which is then used to linearly extrapolate to a fourth point. After the fourth root has been found, the routine predicts the next root by quadratically extrapolating from the three previous roots. After the seventh root, every other root is used for the quadratic extrapolation to reduce the effect of a single solution 'jumping' onto another curve as two modes pass near to each other[1].

USING THE CURVES

The purpose of our program is not just to create dispersion curves; additionally, it is designed to help improve test procedures for complex geometries. To accomplish this purpose, the program incorporates many facilities to extract information from the dispersion curves.

In order to help choose an optimum mode for testing, the program can display the dispersion curves in many forms. Once the dispersion curves are found in frequency, real wave number, and attenuation space, they can easily be manipulated to show information such as phase velocity, group velocity, and angle of incidence. Combinations of these are also helpful, for example a three dimensional plot of frequency versus group velocity versus attenuation allows the user to choose a point on a mode that is both fast enough that it will be well separated in time from the other modes and non-attenuative enough that it will propagate long distances.

The program also helps indicate how to generate a selected mode. Angle of incidence plots simplify the use of angle-beam transducers by showing directly what angle needs to be used. Other program options aid the design of interdigital (or 'comb') transducers[9]. The danger of exciting multiple modes can be easily indentified.

The characteristics of a proposed mode can also be examined. A mode shape display continually updates profiles of the stress, displacement, and energy through the thickness of the system as points are selected on a dispersion curve; this option enables the user to intuitively compare the properties of different modes and study how the properties change along a mode The mode shape calculation takes advantage of the generality of the global matrix method. By

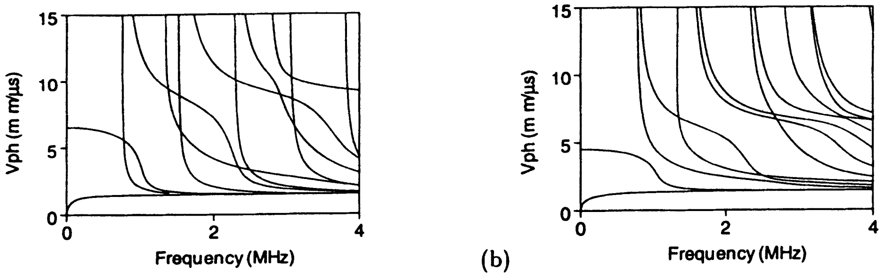


Figure 5. Phase velocity curves for a 1 mm $[0,90]_S$ graphite reinforced epoxy composite (a) parallel to the outer fibers and (b) at an angle of 45 degrees to the outer fibers

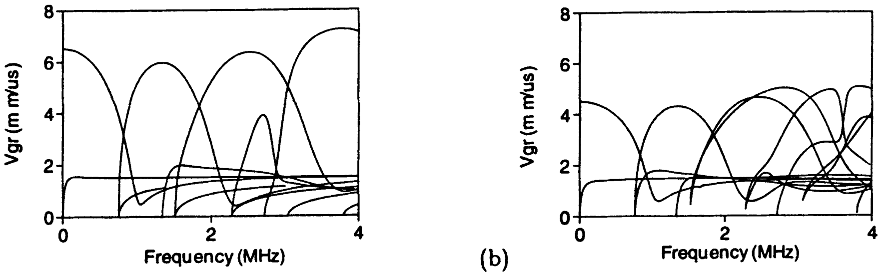


Figure 6. Group velocity curves for a 1 mm $[0,90]_S$ graphite reinforced epoxy composite (a) parallel to the outer fibers and (b) at an angle of 45 degrees to the outer fibers

arbitrarily assuming a value for the amplitude of the longitudinal partial wave in the second layer, the comparative amplitudes of the rest of the partial waves can be extracted. Once the wave amplitudes are known, the layer matrix is used to calculate the values for the stresses and the displacements. An interface to our Finite Element program allows the these mode shapes to be used to define the input when modelling the interaction of guided waves with defects.

EXAMPLES

Two sets of examples follow to demonstrate some of the capabilities of the program.

The first set of examples demonstrates guided wave propagation in composite plates for two different propagation directions. The example shown in Figure 5 is a one millimeter thick $[0,90]_S$ graphite reinforced epoxy plate in vacuum. The system was modelled as three transversely isotropic layers. The material constants for a unidirectional graphite reinforced epoxy, given in Table I, were rotated to the appropriate orientations before creating the layer matrices and assembling the global matrix. Part (a) shows the dispersion curves for waves that are propagating parallel to the fibers in the outer layers and part (b) shows the curves for waves that propagate at an angle of 45 degrees to the direction of the fibers in the outer layers. The group velocity curves in Figure 6 illustrate that the modes travel more quickly when they are aligned with the outer fibers than they do when they are at a skewed angle to them and hint at the complexity of this system.

The second, more extensive, example examines two variations of a steel bar. Figure 7 shows the dispersion curves for a relatively simple system, a 10 mm steel bar in water, which has previously been studied by many authors such as Nagy[10]. As the modes propagate down the bar, energy leaks into the surrounding liquid and the mode attenuates. However, there are certain points, such as the point marked (b) on Figure 7(b) where the attenuation becomes zero These points correspond to the areas where there is no normal displacement (u_r) on the surface

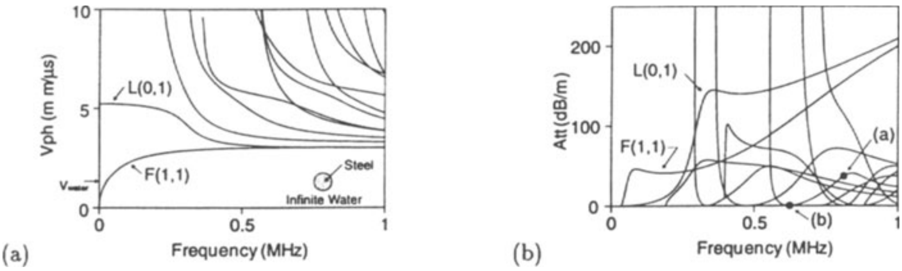
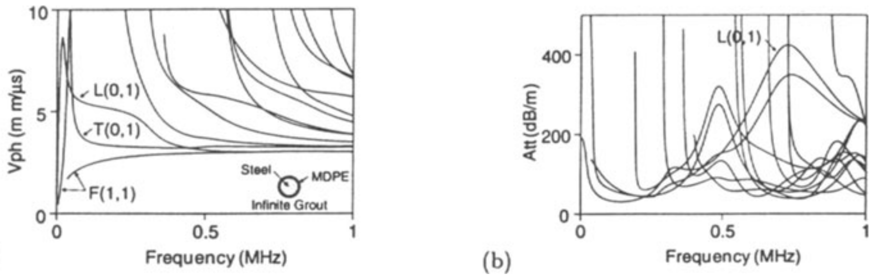


Figure 7. Dispersion Curves for a 10 mm steel bar immersed in water, zero and one order modes, (a) phase velocity and (b) attenuation.

Figure 8. Normal(u_r) and tangential(u_z) displacements at positions marked in Figure 7(b) for a 1 cm steel bar in water. The mode shape in part (a) corresponds to an attenuative point and (b) to a non-attenuative point.

of bar. Thus energy cannot be transmitted into the fluid and the mode travels in the bar unattenuated. Figure 8 shows the mode shapes for two typical points on the dispersion curve. At point (a), there is a strong normal displacement component at the surface of the bar (at the position of 5 mm). This displacement allows the wave to couple into the surrounding water where it propagates away. At point (b), the energy is not coupled into the fluid since there is no normal displacement on the surface of the bar. The discontinuity of the tangential displacement at the boundary between the two layers agrees with the assumption that a liquid-solid boundary only couples normal motion and stress, not tangential or shear components. There is also a section of the first flexural mode (F(1,1)) that does not leak energy. When the phase velocity of a mode is below the bulk velocity of the surrounding liquid, an inhomogeneous wave is formed in the water that decays exponentially with increasing radius. The energy is trapped close to the boundary and travels along the interface.

Figure 9 shows the dispersion curves for a similar, but more complicated system than that shown in the previous figure. The ten millimeter diameter steel bar was coated with a one millimeter layer of medium density polyethylene (MDPE) and placed in an infinitely large block of grout so we could model the ultrasonic inspection of a plastic coated reinforcement bar that is being proposed to improve corrosion resistance in reinforced concrete structures. As indicated in Table I of material constants, both MDPE and grout are very attenuative visco-elastic isotropic materials. At high frequencies, the phase velocity curves are very similar to the case of a steel bar in water, however there are many significant differences, especially for the fundamental modes that we are most interested in for our ultrasonic tests. The first flexural mode (F(1,1)), shown at the bottom of the phase velocity graph, splits into two sections, one of which begins at zero phase velocity and the other which begins at the shear bulk velocity of the grout half-space. The phase velocity of the first torsional (T(0,1)) mode no longer coincides with the bulk shear velocity of the steel bar. The coupling on the outside of the bar does not allow the bar to freely rotate and as the tangential displacements are converted into radial and axial displacements the wave number of the mode must change to maintain the boundary conditions. The coupling also causes the first longitudinal mode (L(0,1)) to no longer tend to the bar velocity ($\sqrt{E/\rho}$). Instead, the phase velocity of this mode changes rapidly at low frequency and tends to zero phase velocity and zero frequency. The attenuation values for this more complicated case also change dramatically. For the simple case of a steel bar in water, the attenuation of the first



(a) (b)

Figure 9. Dispersion curves for a 1 cm steel bar coated with 1 mm MDPE and embedded in grout, zero and one order modes, (a) phase velocity and (b) attenuation

Table I. Material constants used in examples.

Material	ρ (g/cm^3)	E_2 (GPa)	E_3 (GPa)	ν_{13}	ν_{23}	G_{13} (GPa)
GRE	1.605	126.6	8.7	0.32	0.50	3.7

Material	ρ (g/cm^3)	v_L $mm/\mu s$	α_L (np/λ)	v_T $mm/\mu s$	α_T (np/λ)
Steel	7.932	5.96	0.0	3.26	0.0
MDPE	0.952	2.344	0.0285	0.953	0.143
Grout	1.60	2.81	0.043	1.70	0.1
Water	1.0	1.483	0.0	-	-

flexural and first longitudinal modes increases linearly at high frequencies. However, the visco-elastic and coupling properties of the coated bar cause the attenuation values to rise and fall as the frequency is increased. In addition, there are no points on the dispersion curves for the coated bar that are non-attenuative. Since both shear and longitudinal waves couple into the surrounding medium the system is much more lossy. Even in a region that would be non-lossy for the first case (which is completely elastic), the visco-elastic damping attenuates inhomogeneous waves as they travel, causing significant attenuation.

FUTURE WORK

The development of the program is ongoing and new options are added as our research requires them. Some areas of likely expansion are support for cylindrical transversely isotropic materials and a more efficient orthotropic Cartesian routine that will allow us to calculate leaky anisotropic cases. We also intend to complete our adaptation of the program from X-windows to Microsoft Windows in the near future.

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